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On the stability of high-Reynolds-number flows with closed streamlines

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In steady, two-dimensional, inertia-dominated flows it is well known that the vorticity is constant along the streamlines, which, in a bounded domain, are necessarily closed. For inviscid flows, the variation of vorticity across the streamlines is arbitrary, while for forced, weakly dissipitative flows, it is determined by the balance between viscous diffusion and the forcing. This paper discusses the linear stability of flows of this type to two-dimensional disturbances. Arnol'd's stability theorems are discussed. An alternative functional to Arnol'd's is found, which gives the same stability criteria and which permits a representation of the problem in terms of a Schrödinger equation. Conditions for stability are derived from this functional. In particular it is shown that total flow reversals are potentially unstable. The results are illustrated with respect to the geometrically simple case when the streamlines are circular and the forcing is due to a rotating magnetic field, for which case the stability regions are calculated as a function of two parameters. It is shown that the entire theory, including Arnol'd's theorems, applies also to poloidal axisymmetric flows.

1. Introduction

We consider steady, laminar, two-dimensional flow of an incompressible Newtonian fluid in a compact, simply connected domain V bounded by a surface S. The streamlines of such a flow are necessarily closed and may be described by a stream function ψ, so that in Cartesian coordinates the velocity is given by $\boldsymbol{u} = \boldsymbol{\nabla} \wedge (0, 0, \psi)$, and the vorticity by $\boldsymbol{\omega} = (0, 0, \omega)$ where

$$\omega = -\nabla^2\psi \quad \text{in} \quad V; \qquad \psi = 0 \quad \text{on} \quad S. \tag{1.1}$$

When viscous forces and rotational forcing can be neglected, the vorticity must be constant along streamlines

$$\omega = \omega(\psi). \tag{1.2}$$

In this paper, we shall mainly be concerned with unforced inviscid flows, for which the function $\omega(\psi)$ is arbitrary and determined by some initial condition. We shall assume that the flow (ω, ψ) is given and address the question of its stability to small disturbances. We shall confine ourselves almost invariably to the discussion of two-dimensional perturbations, so that the vorticity remains in the z-direction. Such disturbances are of great importance in the geophysical context, where the effects of stratification and especially rotation constrain the flow to quasi-two-dimensionality (e.g. Blumen 1968; McIntyre & Shepherd 1987). These disturbances are also important in unconstrained two-dimensional flow, although often other disturbances dominate (for example, Taylor vortices in circular Couette flow). Occasionally it will be possible to infer the behaviour of more general disturbances from the two-dimensional theory.

The method we shall adopt is a global one, and is based upon the work of Arnol'd (1965, 1966). We derive a variational principle which, if satisfied, guarantees the stability of the flow to two-dimensional disturbances. As is usually the case with global approaches, failure to satisfy the stability criteria does not in itself imply instability. To obtain the genuine stability boundary one must usually resort to normal mode techniques. Such an approach is, however, hard enough even for the geometrically simple case when all the streamlines are circular, as in the Taylor (1923) problem. For the general case, with arbitrarily shaped streamlines, this approach is impractical. Global criteria can however be found which, when satisfied, rigorously guarantee stability, and which, when not, provide a useful indication of the likelihood of the onset of instability. The lack of any alternative results whatsoever increases greatly the value of global results for many practical problems.

Arnol'd's stability theory relies upon various invariants of inviscid flows and cannot therefore be applied blindly to dissipative systems. However, in some circumstances, discussed in a companion paper (Mestel 1989), flows that are both forced and viscous may be described by (1.2) together with a relation expressing the balance of the forcing and viscous diffusion, namely

$$\frac{\mathrm{d}\omega}{\mathrm{d}\psi}=\frac{\oint \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{l}}{\oint \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}}, \tag{1.3}$$

where $\boldsymbol{F}$ is the (rotational) driving force per unit viscosity. The integrals in (1.3) are taken around the closed streamlines ψ = constant. We shall find that $\mathrm{d}\omega/\mathrm{d}\psi$ is an important quantity in our stability calculations and shall therefore occasionally refer to (1.3) for physical interpretation of this derivative. To some extent, we may regard inviscid flows as being driven by some fictitious force against the weak viscous resistance. Nevertheless, the results of inviscid stability calculations can only be applied with great caution to such flows, even in the absence of boundary layers.

In the next section, we review the linear version of Arnol'd's theory as used by Andrews (1983). In §3 we derive a variational principle involving a functional which, while not itself conserved by the flow, is easier to use than Arnol'd's functional and which gives identical linear stability criteria. This variational principle corresponds to a Schrödinger eigenvalue equation. Using this form of the stability criterion, we derive in §4 various simple relations which may be used as stability tests for a general flow satisfying (1.2). In §5 we prove that a certain class of flow reversals are always potentially unstable. In §6 we illustrate these results by considering the simple case when the surface S is circular. In particular, we consider the flow in an electromagnetically stirred cylindrical column of liquid metal. The general results derived in this paper are compared with the precise stability conditions obtained from normal mode analysis of circular flows. Finally, in §7 we discuss the extension of these results to poloidal, axisymmetric flow configurations. It is shown that the entire two-dimensional theory carries over into axisymmetry with but minor modifications. We conclude in §8.

2. Arnol'd's stability theorems

We consider a given two-dimensional region V with a basic steady flow described by $\omega = \omega(\psi)$. We shall also describe the flow by $\psi = \psi(\omega)$. If $\omega(\psi)$ is not a monotonic function then some care will have to be taken to ensure that the correct values of ω

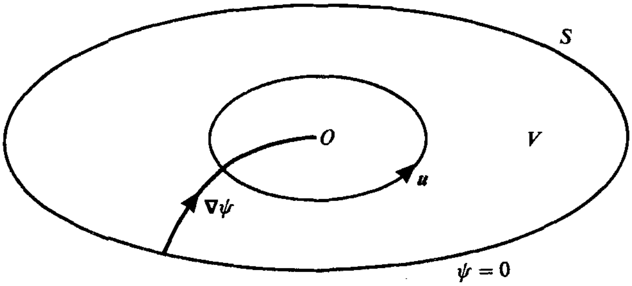

FIGURE 1. A topologically simple streamline pattern.

and ψ are used in any expression. For reference, it is useful to think of a topologically simple streamline pattern, such as in figure 1. For simplicity, we assume that V is simply connected, so that it is bounded by a connected surface S on which $\psi = 0$. Nevertheless, most of our results will hold with but minor modification for domains V with a more complex topology (for example, regions with an inner as well as an outer boundary). A further generalization would be to include in the base flow a velocity component in the z-direction which is constant on any individual streamline, leading to helical rather than closed streamlines. Although such an extension would be straightforward, we shall consider here flows that are strictly two-dimensional.

When the steady flow is perturbed by an unsteady but two-dimensional disturbance it may be represented by means of a stream function $\Psi(x,y,t)$

$$\Psi(x,y,t) = \psi(x,y) + \epsilon\phi(x,y,t), \tag{2.1}$$

where ϵ is some small parameter, and ϕ vanishes on the boundary S so that Ψ does also. The behaviour of the flow is governed by the vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(-\nabla^2\Psi) = 0, \tag{2.2}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative following the unsteady fluid motion. Arnol'd observed that for any such flow, the functional

$$A(\Psi) = \int_V [\tfrac{1}{2}|\nabla\Psi|^2 - C(-\nabla^2\Psi)]\,\mathrm{d}V \tag{2.3}$$

is conserved by the flow for any function C, i.e.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0. \tag{2.4}$$

This follows from the individual conservation laws for kinetic energy and for any function of the vorticity. Now to lowest order in ϵ,

$$A(\Psi) = A(\psi) + \epsilon\int_V [\nabla\psi\cdot\nabla\phi + \nabla^2\phi\, C'(-\nabla^2\psi)]\,\mathrm{d}V. \tag{2.5}$$

Using the divergence theorem, we see that the functional $A(\Psi)$ will be stationary when $\Psi = \psi$ if the function C is chosen so that

$$C'(\omega) = \psi(\omega). \tag{2.6}$$

Then
$$A(\Psi) = A(\psi) + \tfrac{1}{2}\epsilon^2 B(\phi) + o(\epsilon^2), \tag{2.7}$$

where the functional $B(\phi)$ is given by

$$B(\phi) = \int_V \left[|\nabla\phi|^2 - \frac{d\psi}{d\omega} (\nabla^2\phi)^2 \right] dV. \tag{2.8}$$

Now $A(\Psi)$ is conserved by the flow. It follows therefore that for all time $B(\phi)$ is equal to its initial value in the linear approximation. The magnitude of the perturbation ϕ can only increase if it contrives to keep the value of $B(\phi)$ small. Thus, an unstable normal mode $\phi = \Phi(x,y)\,e^{\sigma t}$, where $\mathrm{Re}(\sigma) > 0$, must satisfy $B(\Phi) = 0$. There are many dynamical constraints on the function ϕ, which must satisfy (2.2) to order ϵ. From now on, however, we relax totally these constraints. Should the basic flow (ω, ψ) and domain V be such that $B(\phi)$ is bounded away from zero for *all* functions $\phi(x,y,t)$, then this will certainly also be the case for the smaller class of dynamically permissible ϕ. Whatever then the initial perturbation, it will be unable to grow in magnitude. This proves, in essence, Arnol'd's stability theorems. The flow is stable if, for all non-trivial functions ϕ vanishing on S, either

$$B(\phi) > k\|\phi\| > 0 \tag{2.9}$$

or

$$B(\phi) < -k\|\phi\| < 0 \tag{2.10}$$

for some positive constant k and some suitable norm on ϕ, say

$$\|\phi\| = \left[\int_V |\nabla\phi|^2\, dV \right]^{\frac{1}{2}}. \tag{2.11}$$

In fact Arnol'd proved a stronger result. Slight modification of the functional B, so as to avoid the use of a Taylor expansion, suffices to prove *nonlinear* stability of the flow, if either (2.9) or a stronger version of (2.10) holds. In this paper we are only interested in the linear theory, and so formulate the results as above (cf. Andrews 1983).

Let us now consider the functional $B(\phi)$ and ask under what circumstances it can be of definite sign. Clearly, if

$$\frac{d\psi}{d\omega} \leqslant 0 \text{ throughout } V \tag{2.12}$$

then (2.9) holds, and the flow is stable. This is Arnol'd's first theorem. Now suppose that $d\psi/d\omega > 0$ in some subregion, V_-, of V. It is intuitively obvious that $B(\phi)$ can be made negative by judicious choice of ϕ. For if ϕ is made to vary on very short lengthscales within V_- and on much larger ones outside V_-, then the high derivatives in V_- of the second term will dominate the integral in (2.8) which will therefore be negative. A similar argument shows that if $d\psi/d\omega < 0$ in some region V_+ then $B(\phi)$ can be made positive. If non-empty regions V_- and V_+ exist simultaneously then it will be possible by continuity to find functions ϕ for which $B(\phi) = 0$. Such a ϕ might, in principle, grow. Thus if (2.12) does not hold everywhere the only possibility for further application of Arnol'd's theory is when

$$\frac{d\psi}{d\omega} \geqslant 0 \quad \text{throughout } V. \tag{2.13}$$

When this occurs, the two terms in (2.8) are in competition. We have already observed that the term with the higher derivatives can be made the greater of the two. Thus, for some suitable norm on ϕ, the functional $B(\phi)$ has no lower bound.

However, a global maximum for B will exist. Arnol'd's second theorem (2.10) will apply if this maximum is negative.

Let us therefore seek this maximum by perturbing $B(\phi)$ subject to the constraint

$$\|\phi\|^2 = \int_V |\nabla\phi|^2 \, \mathrm{d}V = 1. \tag{2.14}$$

A similar process was considered by Moffatt (1986). This constraint is to some extent arbitrary and does not affect the final result (2.17). The stationary values of B occur when for some eigenvalue λ

$$\lambda\phi + \frac{\mathrm{d}\psi}{\mathrm{d}\omega}\nabla^2\phi = 0 \quad \text{in} \quad V; \qquad \phi = 0 \quad \text{on} \quad S. \tag{2.15}$$

For an eigenfunction satisfying (2.15), B takes the value

$$B(\phi) = (1-\lambda)\int_V |\nabla\phi|^2 \, \mathrm{d}V. \tag{2.16}$$

The Poincaré inequality

$$B(\phi) \leqslant (1-\lambda_0)\int_V |\nabla\phi|^2 \, \mathrm{d}V \tag{2.17}$$

where λ_0 is the least eigenvalue of (2.15), follows. Thus B is negative definite, and Arnol'd's second theorem applies, provided $\lambda_0 > 1$.

Let us now consider a sequence of flows $\omega(\psi; a)$ varying continuously with some parameter a. If B is negative definite for some range of this parameter, then we know that the flow will be stable for this range. At some particular value $a = a_0$, however, B may attain a maximum value of zero. The stability theorem will then no longer apply, and the flow becomes potentially unstable. At this changeover between flows that are definitely stable and those that may be unstable we know that there must exist an eigensolution to the problem

$$\nabla^2\phi + \frac{\mathrm{d}\omega}{\mathrm{d}\psi}(\psi; a_0)\,\phi = 0 \quad \text{in} \quad V; \qquad \phi = 0 \quad \text{on} \quad S. \tag{2.18}$$

Although we cannot be certain whether or not this changeover indicates a genuine onset of instability, it is of interest to note that the linearized vorticity equation (2.2) may be written as

$$\frac{\partial}{\partial t}\nabla^2\phi + \boldsymbol{u}\cdot\nabla\left[\nabla^2\phi + \frac{\mathrm{d}\omega}{\mathrm{d}\psi}\phi\right] = 0. \tag{2.19}$$

Thus when (2.18) holds for some ϕ, that ϕ is a normal mode for the problem with a zero growth rate. The breakdown of Arnol'd's theory may therefore (but need not) correspond to a genuine onset of instability.

3. An alternative stability functional

Consider now the functional $J(\phi)$ defined by

$$J(\phi) = \int_V \left[|\nabla\phi|^2 - \frac{\mathrm{d}\omega}{\mathrm{d}\psi}\phi^2\right] \mathrm{d}V. \tag{3.1}$$

This functional has no upper bound and is clearly positive definite when (2.12) holds. It attains its minimum subject to the constraint

$$\|\phi\|^2 = \int_V |\nabla\phi|^2 \, \mathrm{d}V = 1 \tag{3.2}$$

when for some eigenvalue λ

$$\frac{1}{\lambda}\nabla^2\phi+\frac{\mathrm{d}\omega}{\mathrm{d}\psi}\phi = 0 \quad \text{in} \quad V; \qquad 0 \quad \text{on} \quad S \tag{3.3}$$

has a solution. For such an eigenfunction,

$$J(\phi) = \left(1-\frac{1}{\lambda}\right)\|\phi\|^2. \tag{3.4}$$

J is thus positive definite provided the lowest eigenvalue λ of (3.3) is greater than unity. Comparison of (3.3) and (2.15) demonstrates that the conditions for Arnol'd's stability theorems to apply are identical to those for the functional J to be positive definite, provided $\mathrm{d}\omega/\mathrm{d}\psi$ is single signed.

From a practical viewpoint this is a very useful result. The functional J is much easier to handle than is B. Furthermore, the variational principle for J being positive may be represented as a Schrödinger equation, as we shall see later. We may thus draw on the wealth of theory on this equation to derive some simple bounds on the stability regions.

Unlike the Arnol'd invariant B, J is not necessarily conserved during fluid motion. Although it may also be expressed as the second variation of a functional that is stationary when $\Psi = \psi$, namely

$$I(\Psi) = \int_V [\tfrac{1}{2}|\nabla\Psi|^2 - D(\Psi)]\,\mathrm{d}V, \tag{3.5}$$

where

$$D'(\psi) = \omega(\psi), \tag{3.6}$$

it may (and will) vary in time as $D(\Psi)$ varies. Nevertheless, as we have shown above, if B is single signed (and the flow is stable), then J must be positive and $I(\psi)$ is a local minimum. We may therefore regard I as an 'energy-like' functional. A steady flow is stable if it minimizes I locally. However, while it is true that any unstable normal mode of the system, $\Phi(x,y)\,\mathrm{e}^{\sigma t}$, must satisfy $B(\Phi) = 0$, it need not satisfy $J(\Phi) = 0$.

In the rest of this paper we shall use the positive definiteness of J as our criterion for stability. We have shown that when $\mathrm{d}\omega/\mathrm{d}\psi$ is single signed and $J(\phi) > 0$ for all ϕ, then the flow must be stable. When conversely $J(\phi) < 0$ for some ϕ then the flow is merely 'potentially unstable'. We shall refer to this state as p-unstable to distinguish it from flows that are known to be unstable. It is generally the case with global approaches to stability that the precise onset of instability cannot be determined. In our case the ambiguity arises because we are including in our calculations many perturbations ϕ that are dynamically inaccessible.

It is instructive to note that for one-dimensional flow $\boldsymbol{u} = (u(y), 0, 0)$, equation (2.12) reduces to Rayleigh's inflection-point result

$$u\frac{\mathrm{d}^2u}{\mathrm{d}y^2} \leqslant 0 \tag{3.7}$$

(or, after some modification, to Fjørtoft's 1950 theorem: McIntyre & Shepherd 1987). Correspondingly, for circular flow (2.12) implies

$$u\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru) \geqslant 0, \tag{3.8}$$

where $\boldsymbol{u} = (0, u(r), 0)$ in terms of cylindrical coordinates (r, θ, z). Thus, if $u(r)$ is single signed ($u(r) \geqslant 0$ without loss of generality) we obtain Rayleigh's (1880) inflection-point result for circular flows, namely that a necessary condition for instability is that the vorticity gradient $\mathrm{d}\omega/\mathrm{d}r$ should change sign. Equation (2.12) may thus be considered as the generalization of Rayleigh's criterion to a system with arbitrarily shaped streamlines satisfying (1.2). Interestingly, in the single-signed circular case, this criterion suffices to prove stability against axisymmetric disturbances also. For if we divide (3.8) by u and integrate between 0 and r we obtain

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru) \geqslant 2\lim_{r\to 0}\left(\frac{u}{r}\right) \geqslant 0 \tag{3.9}$$

as $u \geqslant 0$. Thus Rayleigh's *circulation* criterion also holds everywhere and stability to axisymmetric disturbances follows. We shall comment further on the circular case in §6. It is intriguing that a theory based solely on two-dimensional disturbances should have three-dimensional implications, and it is natural to enquire to what extent similar results hold for non-circular flows. Bayly (1988) has shown for two-dimensional flows with convex streamlines that a generalization of Rayleigh's circulation criterion is a sufficient (but not necessary) condition for instability to three-dimensional disturbances. That is, the flow is unstable to disturbances analogous to Taylor vortices if the circulation round a streamline decreases outwards in some region. Now it is easy to show that outward-decreasing circulation occurs if and only if regions of both positive and negative vorticity exist within some streamline. Without loss of generality, let the vorticity be positive at the central stagnation point, O. Analogously to the circular case, the flow may be deemed 'single signed' when the stream function, ψ, varies monotonically outwards, as in figure 1. Then following the argument leading to (3.9), we can integrate (2.12) from O to ψ to show that the vorticity is always positive. Thus, the circulation always increases outwards for flows satisfying (2.12). However, as the circulation criterion is not *necessary* for instability for non-circular flows, we are unable to deduce three-dimensional stability in the general case.

The form of the stability functional J suggests that increasing $\mathrm{d}\omega/\mathrm{d}\psi$ leads to an increased likelihood of instability. There are thus grounds for regarding regions of positive $\mathrm{d}\omega/\mathrm{d}\psi$ as encouraging instability, and it is natural to enquire what physical quantity this reflects. We have already commented on the connection with various inflection-point criteria for simple flows. In the general case, we can gain some feel for the process by considering the forced and dissipative problem which is governed by (1.3). Positive $\mathrm{d}\omega/\mathrm{d}\psi$ occurs when the force $\boldsymbol{F}$ has a positive line integral around the streamline in the direction of the flow. Large $\mathrm{d}\omega/\mathrm{d}\psi$ corresponds to a high ratio between the total rate of vorticity generation within a streamline to the total vorticity contained within it. Thus when the circulation around a streamline is relatively weak one might expect that a perturbation would be capable of extracting energy from the forcing with resultant instability.

The role played by $\mathrm{d}\omega/\mathrm{d}\psi$ in the stability criteria together with its connection with physical quantities might lead one to postulate that a stable flow cannot be rendered unstable by decreasing the vorticity gradient $\mathrm{d}\omega/\mathrm{d}\psi$. If such is the case (and we have provided no proof) then we may extend the validity of our stability criterion to include some flows for which $\mathrm{d}\omega/\mathrm{d}\psi$ passes through zero and yet for which J is always positive (although Arnol'd's functional B is of indefinite sign). In what follows we shall not exclude such cases from the theory. It should be borne in mind that stability has not been proven unless $\omega(\psi)$ is monotonic.

The variational formulation of the stability criterion is very useful for attempts to demonstrate p-instability, as it permits the substitution of chosen trial functions ϕ. Equivalently, however, we may formulate the criterion as an eigenvalue problem. A function ϕ for which $J(\phi) < 0$ exists if and only if the eigenvalue problem

$$\nabla^2\phi + \frac{d\omega}{d\psi}\phi = \lambda\phi \quad \text{in} \quad V; \qquad \phi = 0 \quad \text{on} \quad S \tag{3.10}$$

possesses a positive eigenvalue $\lambda = \lambda_0 > 0$. Equation (3.10) is a Schrödinger equation corresponding in elementary quantum mechanics to a particle of suitable mass restricted to V under a potential $-d\omega/d\psi$ with total energy $-\lambda$. Somewhat curiously, the criterion for stability is identical to that for the non-existence of a negative energy state for such a particle. In the next section we shall apply some of the established quantum mechanical techniques to our problem.

4. Bounds on stability regions

In this section we consider the stability criterion in both its variational (3.1) and eigenvalue (3.10) forms and obtain conditions which guarantee (p-in)stability of the flow.

First, let us define the extreme values α and β attained by $d\omega/d\psi$ in V:

$$\alpha \geqslant \frac{d\omega}{d\psi} \geqslant \beta. \tag{4.1}$$

Then writing

$$J_\alpha(\phi) = \int_V [|\nabla\phi|^2 - \alpha\phi^2]\, dV \tag{4.2}$$

and similarly for J_β we see

$$J_\alpha \leqslant J \leqslant J_\beta. \tag{4.3}$$

Thus we may infer stability if $J_\alpha > 0$ for all ϕ, and p-instability if there exists a ϕ such that $J_\beta < 0$. We now consider the equivalent eigenvalue problems appropriate to J_α and J_β.

Let k_0^2 be the smallest eigenvalue of the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0 \quad \text{in} \quad V; \qquad \phi = 0 \quad \text{on} \quad S. \tag{4.4}$$

It is clear that the flow will be stable if

$$\alpha < k_0^2 \tag{4.5}$$

and p-unstable if

$$\beta > k_0^2. \tag{4.6}$$

Equation (4.5) is clearly an improvement on the estimate (2.13) and is related to Arnol'd's second theorem (1966, see also McIntyre & Shepherd 1987). The eigenvalue k_0^2 can be bounded geometrically (Garabedian 1964, p. 409). Provided the domain V is star shaped (i.e. there exists an origin O about which S may be described in polar coordinates by $r = f(\theta)$ where f is single valued) then

$$\frac{1}{2}\oint_S \frac{dl}{h} \geqslant \frac{A' k_0^2}{j_{01}^2} \geqslant \pi, \tag{4.7}$$

where A' is the area of V and h is the distance from O onto the tangent to S. In (4.7) j_{01} is the first zero of the Bessel function J_0 ($k_0^2 = j_{01}^2$ when V is a circle of unit radius). Equation (4.7) may be used in conjunction with (4.5) and (4.6) if desired.

We remarked in §3 on the similarity between our problem and the two-dimensional Schrödinger equation. In the remainder of this section we exploit that analogy. It is clear from the variational formulation that we may restrict the function ϕ at will provided we are seeking sufficient conditions for p-instability. In particular, we may derive a one-dimensional Schrödinger equation which is more amenable to analysis. A natural restriction is to require $\phi = \phi(\psi)$, so that the perturbation does not alter the shape of the streamlines to lowest order. With this restriction

$$J(\phi) = \int_V \left[\left(\frac{d\phi}{d\psi}\right)^2 q^2 - \frac{d\omega}{d\psi}\phi^2\right]\frac{dl\,d\psi}{q}, \tag{4.8}$$

where dl denotes an integral round the closed streamline keeping ψ constant, and $q = |\boldsymbol{u}|$. If we write

$$a(\psi) = \oint_\psi q\,dl, \quad b(\psi) = \oint_\psi \frac{dl}{q}, \quad y(\psi) = \int_0^\psi \frac{d\psi'}{a(\psi')} \tag{4.9}$$

then we obtain

$$J(\phi) = \int_0^\infty \left[\left(\frac{d\phi}{dy}\right)^2 - ab\frac{d\omega}{d\psi}\phi^2\right]dy, \tag{4.10}$$

noting that y is logarithmically infinite as ψ tends to its maximum value at the central stagnation point. The existence of negative values of $J(\phi)$ is equivalent to the existence of positive eigenvalues λ for the equation

$$\frac{d^2\phi}{dy^2} - \mathscr{V}(y)\,\phi = \lambda\phi, \tag{4.11}$$

where the potential

$$\mathscr{V}(y) = -ab\frac{d\omega}{d\psi}. \tag{4.12}$$

We may now apply established methods from elementary quantum mechanics to (4.11). For instance, we may inscribe a 'square-well' potential $\mathscr{V}_0$ inside the actual potential $\mathscr{V}$. Suppose for simplicity that $\mathscr{V}$ is a monotonic increasing function of y (and is therefore negative as $\mathscr{V} \to 0$ as $y \to \infty$). Then we may define for some y_0

$$\mathscr{V}_0(y) = \begin{cases} \mathscr{V}(y_0) & \text{for } 0 < y < y_0 \\ 0 & \text{for } y > y_0, \end{cases} \tag{4.13}$$

so that $\mathscr{V} \leqslant \mathscr{V}_0$ everywhere. The existence of a bound state ($\lambda \leqslant 0$) for $\mathscr{V}_0$ gives as a condition for p-instability

$$\max_{y_0 > 0}\,[-y_0^2\,\mathscr{V}(y_0)] > \tfrac{1}{4}\pi^2 \tag{4.14}$$

when we consider all permissible y_0. The above 'square-well' approximation will give a good bound when the potential $\mathscr{V}$ varies quickly with y. When on the other hand $\mathscr{V}$ is slowly varying, a WKB approximation may be used. The condition for p-instability is then found to be

$$\int_0^\infty \left(ab\frac{d\omega}{d\psi}\right)^{\frac{1}{2}} dy > \frac{3\pi}{4} \tag{4.15}$$

(Murzbacher 1970, chapter 7). Unlike the other conditions we have obtained, we cannot rigorously infer p-instability should (4.15) hold marginally, although the possible error of the approximation could be bounded.

We have thus developed a number of conditions which are of use in determining

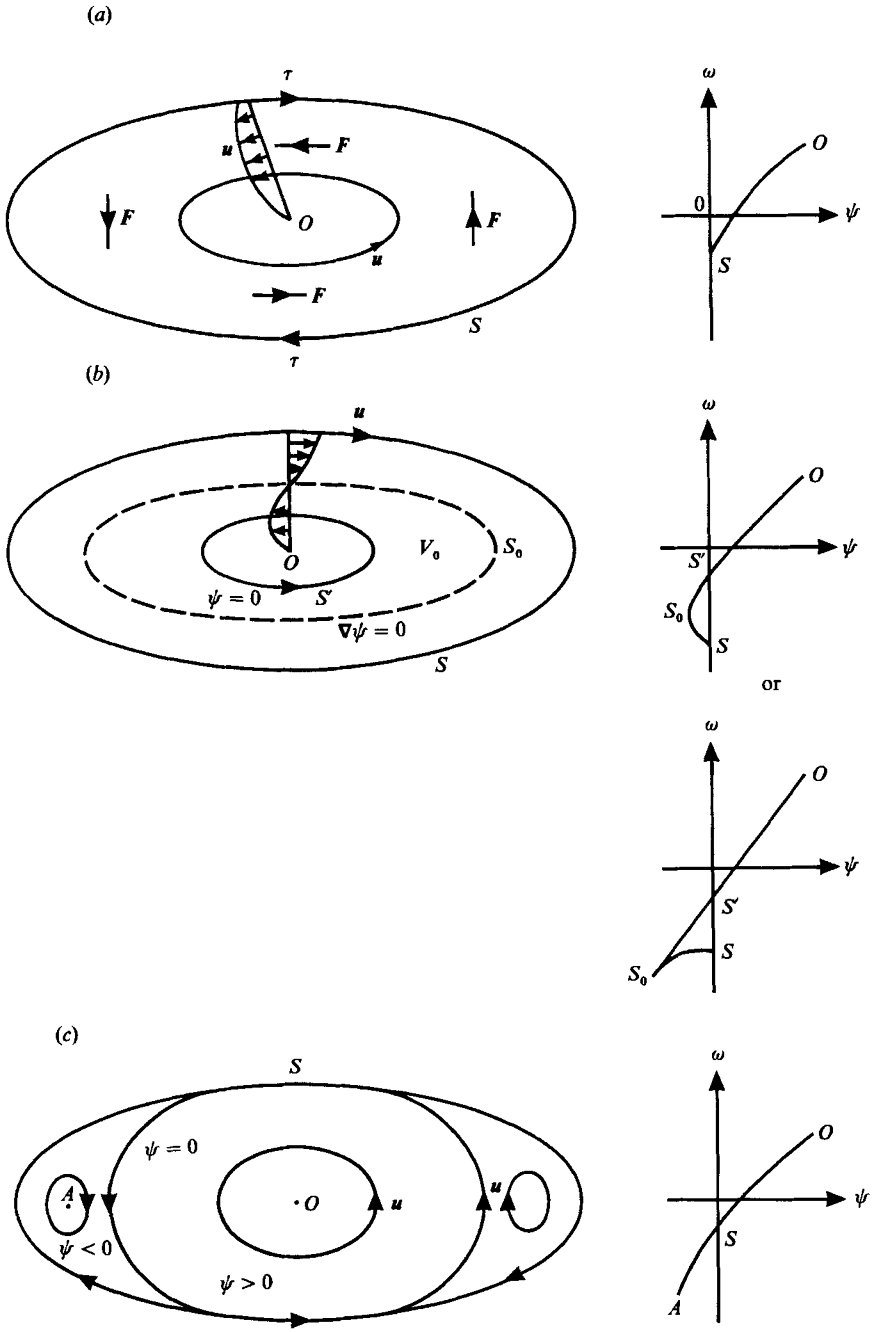

FIGURE 2(a–c). For caption see facing page.

the stability or otherwise of a flow (ω, ψ). We conclude this section with the observation that when $\phi = \psi$, (3.1) and (1.1) imply

$$J(\psi) = \int_V \psi \left[\omega - \frac{\mathrm{d}\omega}{\mathrm{d}\psi}\psi\right] \mathrm{d}V. \tag{4.16}$$

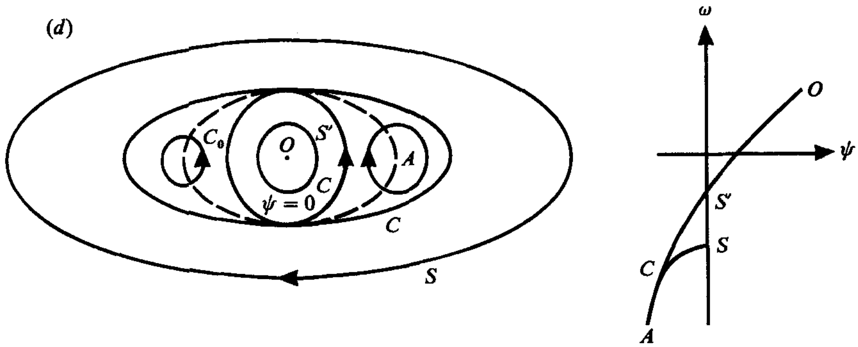

FIGURE 2. Schematic behaviour of a flow driven by an anticlockwise force F and a varying surface stress τ. (a) Unseparated flow for $\tau < \tau_0$; (b) simultaneous separation, shown to be p-unstable, $\boldsymbol{u} = 0$ on S_0; (c) local separation for $\tau_0 < \tau < \tau_1$; (d) separated flow without stagnation streamline $\tau_1 < \tau < \tau_2$.

Thus when $\psi \geqslant 0$ in V, the flow will be p-unstable whenever

$$\frac{d\omega}{d\psi} > \frac{\omega}{\psi} \quad \text{throughout } V. \tag{4.17}$$

In the next section we shall prove a general result concerning the p-instability of flow reversals.

5. Instability of flow reversals

Before we proceed with the statement and proof of the theorem, it is useful to consider the following thought experiment. Suppose we have a flow governed by (1.2) driven by an anticlockwise force $\boldsymbol{F}$ acting in V, and a clockwise tangential stress τ acting on S, as in figure 2. Global equilibrium is maintained by a suitable normal stress on S (provided S is non-circular). Such problems are solved in Mestel (1989). Physically, we would expect that when $|\boldsymbol{F}| \gg \tau/L$, where L is a typical lengthscale of V, a single anticlockwise gyre will form as in figure 2(a), while if $\tau/L \gg |\boldsymbol{F}|$ a single clockwise gyre will form. If we start from the former end, with τ negative, and consider the solution as τ increases towards and through zero, it is clear that as the resistance of S grows, at some value τ_0 the velocity will fall to zero somewhere on S. For $\tau > \tau_0$ a region of reversed flow will develop either as in figure 2(b), if separation occurs simultaneously everywhere on S, or as in figure 2(c), if separation is localized. In the latter case, as τ increases still further, the region of anticlockwise circulation will shrink, separate totally from S at some value τ_1 (figure 2d), and contract to a point when $\tau = \tau_2$. The form of the function $\omega(\psi)$ in the various cases is indicated on the figures. The behaviour shown in figures 2(c) and 2(d) is more general than that of figure 2(b), but the latter is of importance in some special flows (including, but not only, circular flows). More complex behaviour is also possible. We shall show in this section that the topologically simple flow depicted in figure 2(b) must be p-unstable. Formally, we prove the following theorem:

THEOREM. *Given a bounded, two-dimensional, simply connected domain V and an* *analytic flow $\omega(\psi)$ for which there exists a streamline S_0 on which $\nabla\psi = 0$, then that flow is p-unstable, that is*

$$\exists\, \phi \text{ s.t.} \int_V \left[|\nabla\phi|^2 - \frac{d\omega}{d\psi}\phi^2\right] dV < 0, \qquad \phi = 0 \quad \text{on} \quad S. \tag{5.1}$$

We prove the theorem by explicitly exhibiting a perturbation ϕ for which (5.1) holds. Let

$$\boldsymbol{u} = \nabla \wedge (0, 0, \psi) = q\hat{\boldsymbol{\xi}}, \tag{5.2}$$

where $\hat{\boldsymbol{\xi}}$ is a unit vector. We note that

$$\nabla^2 \boldsymbol{u} = -\nabla \wedge (0, 0, \omega) = -\frac{d\omega}{d\psi}\boldsymbol{u}, \tag{5.3}$$

and so

$$\boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} = -q^2 \frac{d\omega}{d\psi}. \tag{5.4}$$

We define ϕ_1 such that

$$\phi_1 = \begin{cases} q & \text{inside } V_0 \\ 0 & \text{outside } V_0 \quad \text{and so } S_0, \end{cases} \tag{5.5}$$

where V_0 is the region inside S_0 as in figure 2(*b*). Then ϕ_1 is continuous everywhere and differentiable except on S_0. Furthermore, we note that ϕ_1 may be 'smoothed out' near S_0 in such a way that $J(\phi_1)$ is changed by an arbitrary small amount. Thus a fully differentiable (analytic) function ϕ_0 may be found such that if $J(\phi_1) < 0$ then $J(\phi_0) < 0$. It thus suffices to prove the former. In addition, since ϕ_1 vanishes on S_0 (where $d\omega/d\psi$ will be singular unless $\nabla\omega = 0$ there), the convergence of the integral is assured. Now

$$\begin{aligned} J(\phi_1) &= \int_V \left[|\nabla\phi_1|^2 - \frac{d\omega}{d\psi}\phi_1^2\right] dV \\ &= \int_{V_0} \left[|\nabla q|^2 - \frac{d\omega}{d\psi} q^2\right] dV \\ &= \int_{V_0} [\boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} - q\nabla^2 q]\, dV \end{aligned} \tag{5.6}$$

from (5.4) and the divergence theorem, the surface term vanishing because $q = 0$ on S_0 and V is simply connected. We now show that the integrand of (5.6) is negative. For

$$\begin{aligned} \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} &= q\hat{\boldsymbol{\xi}} \cdot (\nabla^2 q \hat{\boldsymbol{\xi}} + 2\nabla q \cdot \nabla\hat{\boldsymbol{\xi}} + q\nabla^2 \hat{\boldsymbol{\xi}}) \\ &= q\nabla^2 q + q^2 \hat{\boldsymbol{\xi}} \cdot \nabla^2 \hat{\boldsymbol{\xi}} \end{aligned} \tag{5.7}$$

since $\hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\xi}} = 1$, and hence $\nabla q \cdot \nabla(\hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\xi}}) = 0$. Further,

$$0 = \nabla^2(\hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\xi}}) = 2\hat{\boldsymbol{\xi}} \cdot \nabla^2 \hat{\boldsymbol{\xi}} + 2\frac{\partial \hat{\xi}_i}{\partial x_j}\frac{\partial \hat{\xi}_i}{\partial x_j}. \tag{5.8}$$

Combining (5.6), (5.7) and (5.8) we have

$$J(\phi_1) = -\int_{V_0} q^2 \frac{\partial \hat{\xi}_i}{\partial x_j}\frac{\partial \hat{\xi}_i}{\partial x_j}\, dV \leqslant 0. \tag{5.9}$$

Now equality in (5.9) holds only if either $q = 0$ throughout V_0 (i.e. the fluid is at rest everywhere as q is analytic) or if $\hat{\boldsymbol{\xi}}$ is a constant vector, and the flow is uni-directional.

But this is not possible in a bounded domain. So $J(\phi_1) < 0$ and the theorem is proved.

An obvious corollary of the theorem concerns flow inside a rigid stationary container. Should a flow of the form $\omega = \omega(\psi)$ exist, satisfying a no-slip condition on the boundary, then it will be p-unstable, by the theorem. However it should be noted that not all rigid boundary shapes will permit such a solution. In general, an unstable boundary-layer structure may develop. The circumstances in which a forced, viscous flow can be of the form (1.2) are discussed in Mestel (1989).

We have observed that $\mathrm{d}\omega/\mathrm{d}\psi$ may become infinite on the streamline S_0 where the velocity vanishes. If this occurs, then $\mathrm{d}\omega/\mathrm{d}\psi$ will not be single signed in the interior of V unless S_0 coincides with the boundary surface S. We argued in §2 that if $\mathrm{d}\omega/\mathrm{d}\psi$ takes both signs within V then the Arnol'd invariant A can be of either sign, and hence that J could be negative. The theorem confirms this. In addition, it shows that the flow is p-unstable even when $\mathrm{d}\omega/\mathrm{d}\psi$ is infinite on S but is otherwise single signed (the rigid boundary case), and also when $\boldsymbol{\nabla}\omega$ vanishes on S_0 so that $\mathrm{d}\omega/\mathrm{d}\psi$ is regular there.

The author has not been able to demonstrate p-instability for the type of separation shown in figure 2(*d*). Such a flow is characterized by the existence of a closed curve C_0, drawn dashed in the figure, on which the tangential velocity vanishes. It seems likely that the flow of figure 2(*d*) is also necessarily p-unstable, although, unlike that of figure 2(*b*), it is at least *topologically* stable, in that small disturbances do not alter the streamline structure, as pointed out privately by H. K. Moffatt.

The proof of the theorem illustrates a consequence of the variational formulation (3.1) which is to be physically expected; namely, that if a portion of a flow is p-unstable then so is the entire flow. This idea is of importance because it extends the use of our analysis to a much wider class of flows for which (1.2) is valid only in a sub-region. Thus for example one can conceive of instances in which a complex boundary-layer structure might exist around a solid surface with the flow settling down in its interior to one of type (1.2). Should the sub-flow inside any closed streamline be shown to be unstable, then we may infer that the entire flow is also.

We must now address the question of the relation between p-instability and genuine instability. The only case where comparisons can be made with a normal mode analysis occurs when the streamlines are circular. In the next section we investigate this case.

6. Circular flow driven by a rotating magnetic field

In this section we illustrate the results we have already obtained with respect to one of the few available analytical solutions for flows of type (1.2), which occurs when all the streamlines are circular. For such geometry the inertia forces are exactly balanced by radial pressure gradients and the Navier–Stokes equations effectively linearize. In terms of polar coordinates (r, θ, z) we let S be $r = a$. In $r < a$ a body force $\boldsymbol{F} = \mu(F_r(r), F_\theta(r), 0)$ acts, where μ is the fluid viscosity. The velocity $\boldsymbol{u} = (0, u_\theta, 0)$ then satisfies

$$F_\theta = \left(\nabla^2 - \frac{1}{r^2}\right) u_\theta = -\frac{\mathrm{d}\omega}{\mathrm{d}r} \tag{6.1}$$

or

$$u_\theta = \tfrac{1}{2}\omega_0\, r + \tfrac{1}{2} r \int_r^a F_\theta \,\mathrm{d}r + \frac{1}{2r}\int_0^r r^2 F_\theta \,\mathrm{d}r, \tag{6.2}$$

where ω_0 is the vorticity on S. Equation (1.3) reduces to

$$\frac{\mathrm{d}\omega}{\mathrm{d}\psi} = \frac{F_\theta}{u_\theta}. \tag{6.3}$$

Let us now recall the known stability theory for inviscid circular flows (see for example Drazin & Reid 1981). The flow is stable to two-dimensional disturbances if Rayleigh's inflection-point criterion fails, that is, if throughout V

$$\frac{\mathrm{d}\omega}{\mathrm{d}r} \neq 0. \tag{6.4}$$

Further, a necessary and sufficient criterion for stability to axisymmetric disturbances is that Rayleigh's discriminant should be everywhere positive

$$u_\theta \omega > 0. \tag{6.5}$$

Now we observed in §3 that if the flow is single signed (say $u_\theta \geqslant 0$) then Arnol'd's first theorem (2.12) suffices to prove not only (6.4) but also (6.5). However, if there is a flow reversal as in the last section, it is possible for $\mathrm{d}\omega/\mathrm{d}\psi$ to change sign while $\mathrm{d}\omega/\mathrm{d}r$ is single signed. Thus, as pointed out by a referee, in this case although the flow is p-unstable, it will be stable to two-dimensional disturbances. Clearly, however, flows with a flow reversal violate (6.5) and are thus unstable to axisymmetric disturbances. Indeed, physical intuition would suggest that flow reversals will always be unstable to three-dimensional disturbances even in the non-circular case, and this follows from the work of Bayly (1988), at least when the streamlines are convex. However, we have not investigated general disturbances in this paper, and can merely observe that there appears to be a loose connection between on the one hand, failure of a condition to guarantee two-dimensional stability, and on the other, onset of more general instability. We shall illustrate this further, together with a discussion of the relative values of the criteria we developed in the last few sections with a specific example.

A case of importance in the metallurgical industry occurs when the force F_θ is generated by a rotating magnetic field. F_θ is then given by

$$F_\theta = \frac{2B_0^2}{\mu\mu_0}\frac{1}{r}\left|\frac{I_1[(1+\mathrm{i})\,r/\delta]}{I_0[(1+\mathrm{i})\,a/\delta]}\right|^2, \tag{6.6}$$

where I_0, I_1 are Bessel functions while B_0, μ_0 and δ are the magnetic field, permeability and skin depth respectively (e.g. Moffatt 1965). For simplicity, we non-dimensionalize the velocity so that effectively

$$\frac{aB_0^2}{\mu_0\mu} = 1. \tag{6.7}$$

For all δ, $F_\theta(r)$ is a positive, monotonic increasing function. When $\delta \gg a$, we have

$$F_\theta \sim \frac{r}{a\delta^2}\left[1+O\left(\frac{a^2}{\delta^2}\right)\right], \tag{6.8}$$

while for $\delta \ll a$ we have the exponential 'skin-depth' behaviour

$$F_\theta \sim \frac{2}{a^2}\mathrm{e}^{2(r-a)/\delta}\left[1+O\left(\frac{\delta}{a}\right)+O\left(1-\frac{a}{r}\right)\right]. \tag{6.9}$$

We shall use the flow given by (6.2), (6.6) and (6.7) as an example and plot the stability regions as functions of ω_0 and δ, as predicted by the (inviscid) theory. As we commented in §1 there is some danger in applying this theory to flows that are in reality viscous, especially in the presence of a solid boundary. If however the boundary is one of prescribed stress, then the quasi-inviscid approximation should hold at least for timescales on which viscous diffusion is negligible.

From (3.10), the p-stability boundary ($\lambda = 0$) occurs when a solution exists to the problem

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}+\frac{F_\theta}{u_\theta}\phi = 0; \quad \phi(a) = 0. \tag{6.10}$$

For fixed δ, we may think of this as an eigenvalue problem for ω_0, implicit in $u_\theta(r)$. Now if u_θ has a zero in the interval $(0, a)$ then the conditions of the reversed-flow theorem apply, and we know that the flow is p-unstable. Thus we are interested in values of ω_0 and δ for which (6.10) is regular away from $r = 0$. Equation (6.10) may be solved numerically for general δ, but we may find asymptotic solutions for δ large and small. As $\delta/a \to 0$, we have

$$\phi \sim 1-\mathrm{e}^{2(r-a)/\delta}, \quad \omega_0 \sim -\frac{1}{a}\left[\frac{\delta}{a}-\left(\frac{\delta}{a}\right)^2+O\left(\frac{\delta}{a}\right)^3\right], \tag{6.11}$$

while as $\delta/a \to \infty$,

$$\phi \sim \mathscr{F}\left(\sqrt{2}, -\sqrt{2}; 1; \frac{r^2/a^2}{4\omega_0\delta^2/a+2}\right), \quad \frac{\omega_0\delta^2}{a} \sim -0.09124+O\left(\frac{a^2}{\delta^2}\right), \tag{6.12}$$

where $\mathscr{F}$ is the hypergeometric function.

The stability regions in the $(\omega_0, a/\delta)$-plane may now be found and are plotted in figure 3. For $|\omega_0|$ sufficiently large (which in this simple geometry corresponds to a superposed solid rotation) it is clear from (6.2) and (6.3) that the flow will be stable. As ω_0 increases from $-\infty$ eventually reversed flow develops in the sense of F_θ against the sense of ω_0, which by the theorem is p-unstable. This corresponds to the lowest curve in the figure. The next highest curve plots that value of $\omega_0(\delta)$ appropriate to a solid boundary $S\,(u_\theta(a) = 0)$. This curve must lie in the p-unstable region. The middle curve is based on the Helmholtz estimate (4.6). It is a lower bound for the upper p-stability boundary. The second highest curve is the exact upper p-stability boundary as calculated numerically from (6.10). Finally, the uppermost curve is based on the WKB approximation (4.15) and overestimates this boundary. For this example, both the other Helmholtz estimate (4.5) and the square-well result (4.14) are worse than (4.15), but this need not be the case in general. These have been omitted from the figure for the sake of clarity. For non-circular shapes, the exact p-stability boundary would be much harder to find and so the WKB, square-well and Helmholtz estimates would be more useful.

Let us now compare these results with the genuine stability regions as found from the extensive normal mode analysis of circular flow. For the particular forcing we have chosen, the flow is in fact always stable to two-dimensional disturbances, as (6.4) holds for all ω_0 and δ. The stability of this flow to axisymmetric perturbations has been analysed by Richardson (1974), who calculated the normal modes for the full viscous problem. In our quasi-inviscid approximation it is sufficient to use the Rayleigh criterion (6.5), which predicts instability to axisymmetric disturbances for negative values of ω_0 lying above the lowest curve in the diagram. It is of interest to note that the p-unstable region lies wholly within the region of three-dimensional instabilities, and that the lower p-stability boundary exactly coincides with one boundary of this region.

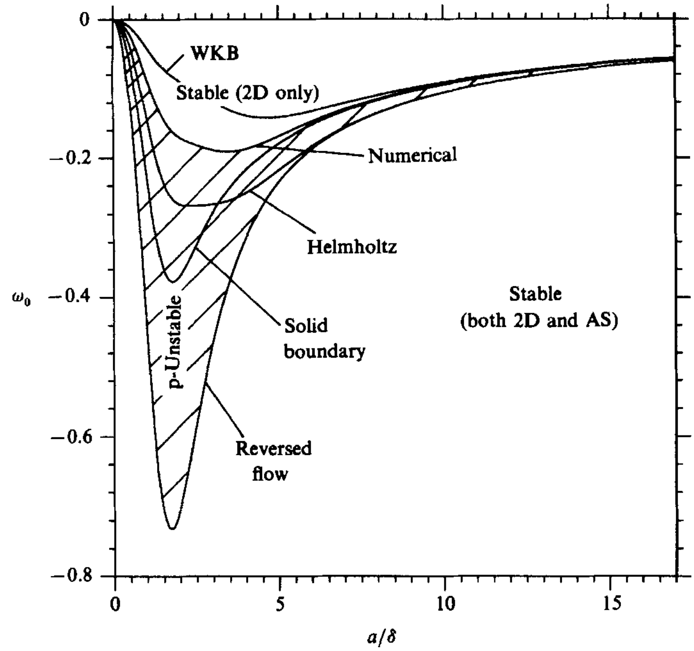

FIGURE 3. Stability regions in the $(\omega_0, a/\delta)$-plane for a rotating magnetic field about a circular cylinder.

Thus, in summary, we can identify four different regions of the $(\omega_0, a/\delta)$-plane for this example. First, there is the region where $\mathrm{d}\omega/\mathrm{d}\psi < 0$ throughout V, for which stability to both two-dimensional (2D) and axisymmetric (AS) disturbances holds, and is proved by Arnol'd's first theorem. Secondly, there is the p-unstable region in which the flow is in fact stable to 2D but unstable to AS disturbances. Thirdly, we have the region where $\mathrm{d}\omega/\mathrm{d}\psi > 0$ and the functional J is positive. In this region, 2D stability is proved, but the flow may or may not be stable to AS disturbances. As we expected, the theory we have developed underestimates the region of 2D stability, but surprisingly, it appears to be connected with more general instability. It is not possible to infer very much about the general case from the study of circular flows. We might however conjecture that p-instability will always occur totally within the regions of parameter space that are unstable to some sort of three-dimensional disturbance. This conjecture is at least plausible if one regards p-instability as highlighting structural weakness in the basic flow. However, as pointed out by a referee, the circular flow $u(r) = r\,\mathrm{e}^{-r^2}$ is stable to both 2D and AS disturbances in the domain $r < 1$, yet it is p-unstable as shown by the trial function $\phi = 1 - r^2$. This conjecture thus appears to be false.

Apart from the final remarks in §8, this concludes the portion of this paper that deals with two-dimensional flows. In the next section we consider to what extent the results we have obtained have counterparts in poloidal, axisymmetric flow.

7. Poloidal axisymmetric flows

The results we have derived so far hold only for two-dimensional flows. It is natural to enquire to what extent analogous results exist for poloidal, axisymmetric flow patterns. In this section we assess the effects of the additional factors of the cylindrical radius which crop up in axisymmetry. The general drift of the analysis is similar to that of §§2–6, and we shall therefore leave out some of the details.

We use cylindrical coordinates (r, θ, z) where $\partial/\partial\theta = 0$, and the volume element $\mathrm{d}V = 2\pi r\,\mathrm{d}r\,\mathrm{d}z$. The Stokes stream function $\psi(r,z)$ is related to the poloidal velocity $\boldsymbol{u}$ by

$$\boldsymbol{u} = \boldsymbol{\nabla}\wedge\left(0, \frac{\psi}{r}, 0\right) = \frac{1}{r}\boldsymbol{\nabla}\psi\wedge\hat{\boldsymbol{\theta}}, \tag{7.1}$$

where $\hat{\boldsymbol{\theta}}$ is a unit vector in the θ-direction. The vorticity is azimuthal and given by

$$\boldsymbol{\omega} = (0, \omega, 0) = -\frac{1}{r}\mathrm{D}^2\psi\hat{\boldsymbol{\theta}} \tag{7.2}$$

where, for any $\eta(r,z)$, the Stokes operator, D^2, satisfies

$$\frac{1}{r}\mathrm{D}^2\eta \equiv \left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\eta}{r}\right) \equiv r\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\eta}{r^2}\right). \tag{7.3}$$

Restricting attention to poloidal axisymmetric perturbations, we can write the unsteady stream function $\Psi(r,z,t)$ in the form

$$\Psi(r,z,t) = \psi(r,z) + \epsilon\phi(r,z,t). \tag{7.4}$$

Here ϕ and Ψ must vanish on the symmetry axis $r = 0$ as well as on S. The unsteady inviscid vorticity equation reduces to a material conservation law for the 'potential vorticity', $-\mathrm{D}^2\Psi/r^2$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(-\frac{\mathrm{D}^2\Psi}{r^2}\right) = 0. \tag{7.5}$$

Thus, the inertial constraint for steady flows corresponding to (1.2) is

$$-\frac{1}{r^2}\mathrm{D}^2\psi = \frac{\omega}{r} = \Omega(\psi). \tag{7.6}$$

The extension of Arnol'd's theory to axisymmetric flows is fairly straightforward. Nevertheless, as it is not well known we shall describe it here. Equation (7.5) ensures the conservation of any function of the potential vorticity integrated over the axisymmetric region V. Kinetic energy is likewise conserved. Thus, equivalent to (2.5), we find that for any function L

$$A_2(\Psi) = \int_V\left[\frac{|\boldsymbol{\nabla}\Psi|^2}{2r^2} - L\left(-\frac{1}{r^2}\mathrm{D}^2\Psi\right)\right]\mathrm{d}V \tag{7.7}$$

is conserved by the flow. To lowest order in ϵ, we have

$$A_2(\Psi) = A_2(\psi) + \epsilon\int_V\left[\frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\phi}{r^2} + \frac{1}{r^2}\mathrm{D}^2\phi\, L'\left(-\frac{1}{r^2}\mathrm{D}^2\psi\right)\right]\mathrm{d}V. \tag{7.8}$$

Using (7.3) and the divergence theorem, we see that the first-order perturbation vanishes if the function L is such that

$$L'(\Omega) = \psi(\Omega). \tag{7.9}$$

Then

$$A_2(\Psi) = A_2(\psi) + \tfrac{1}{2}\epsilon^2 B_2(\phi) + o(\epsilon^2), \tag{7.10}$$

where

$$B_2(\phi) = \int_V\left[\frac{|\boldsymbol{\nabla}\phi|^2}{r^2} - \frac{\mathrm{d}\psi}{\mathrm{d}\Omega}\left(\frac{1}{r^2}\mathrm{D}^2\phi\right)^2\right]\mathrm{d}V. \tag{7.11}$$

Once again, we may infer stability of the flow provided the functional B_2 is of definite sign. Clearly, such is the case if $\mathrm{d}\psi/\mathrm{d}\Omega \leqslant 0$ throughout V. Also, as before, if $\mathrm{d}\psi/\mathrm{d}\Omega$ can take either sign in V, then so can the function B_2. When however $\mathrm{d}\psi/\mathrm{d}\Omega \geqslant 0$ throughout V then B_2 may be negative definite. The critical case occurs when there is an eigensolution, ϕ, to the problem

$$\frac{\mathrm{d}\psi}{\mathrm{d}\Omega}\frac{1}{r^2}\mathrm{D}^2\phi+\phi = 0 \quad \text{in} \quad V; \qquad \phi = 0 \quad \text{on} \quad S. \tag{7.12}$$

Such a ϕ would, as before, be a steady solution to the normal mode equation

$$\frac{\partial}{\partial t}\left(\frac{1}{r^2}\mathrm{D}^2\phi\right)+\boldsymbol{u}\cdot\boldsymbol{\nabla}\left[\frac{1}{r^2}\mathrm{D}^2\phi+\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\phi\right] = 0 \tag{7.13}$$

obtained from (7.5). Similar arguments to those of §3 lead us to the alternative functional J_2 defined by

$$J_2(\phi) = \int_V\left[\frac{|\boldsymbol{\nabla}\phi|^2}{r^2}-\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\phi^2\right]\mathrm{d}V. \tag{7.14}$$

The condition for J_2 to be positive definite is precisely that for B_2 to be of definite sign, provided $\mathrm{d}\Omega/\mathrm{d}\psi$ is single signed. The associated Schrödinger eigenvalue equation for (7.14) is

$$\mathrm{D}^2\phi+r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\phi = \lambda\phi. \tag{7.15}$$

An 'energy-like' functional I_2 may be found here also, namely

$$I_2(\Psi) = \int_V\left[\frac{|\boldsymbol{\nabla}\phi|^2}{2r^2}-P(\Psi)\right]\mathrm{d}V \quad \text{where} \quad P'(\psi) = \Omega(\psi). \tag{7.16}$$

Stability follows if the basic flow (Ω, ψ) is a local minimum of I_2.

Stability estimates based on bounding $r^2\,\mathrm{d}\Omega/\mathrm{d}\psi$, and involving the lowest eigenvalue of the axisymmetric Helmholtz equation in V may be found in a manner identical to that of §4. A one-dimensional Schrödinger equation may be found here also by restricting $\phi = \phi(\psi)$. In fact we obtain the identical equation (4.10) by this approach (with Ω replacing ω). Thus the WKB and 'square-well' results carry straight over. The reversed-flow theorem of §5, however, does not carry over unmodified. We can show that the flow must be either p-unstable or neutrally stable. For if we let $\phi = \partial\psi/\partial z$ inside the region V_0 on whose surface $\boldsymbol{u}$ vanishes, then we can show that

$$J_2\left(\frac{\partial\psi}{\partial z}\right) = 0. \tag{7.17}$$

Now were this value of ϕ not to satisfy (7.15) with $\lambda = 0$, we could deduce that a function ϕ existed for which $J_2(\phi) < 0$ and thus that the flow was p-unstable. However, since

$$\nabla^2\boldsymbol{u} = -\boldsymbol{\nabla}\wedge(\omega\hat{\boldsymbol{\theta}}) = -2\Omega\hat{\boldsymbol{z}}-r^2\frac{\mathrm{d}\Omega}{\mathrm{d}\omega}\boldsymbol{u}, \tag{7.18}$$

or taking the r-component

$$-\frac{1}{r}\mathrm{D}^2\left(\frac{\partial\psi}{\partial z}\right) = r\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\frac{\partial\psi}{\partial z}, \tag{7.19}$$

we see that $\partial\psi/\partial z$ is indeed an eigenfunction of (7.15), and no such deduction is possible. The author has not been able to prove p-instability in the general case, although it seems likely. Of course, if, as before, $d\Omega/d\psi$ is infinite on a streamline in the interior of V, then it will take both signs there, and we know from the form of B_2 that the flow will then be p-unstable.

Analytic solutions for non-constant Ω in axisymmetry are rare, there being nothing analogous to the circular-streamline case of §6. Thus, we do not present an analytic illustration here. However, solutions could be found, and their stability determined, for the special case when the forcing is such that $d\Omega/d\psi$ is constant in regular shape S (such as in Jones, Moore & Weiss 1976).

8. Concluding remarks

In this paper we have limited our attention to a well-defined class of flows with which we can make considerable progress. The stability criterion we have derived and used extensively gives a convenient, global condition for deciding whether a flow of this type is linearly stable. However, the method has its limitations, and it is well to bear these in mind.

First and most importantly, we have no guarantee that a flow deemed p-unstable by the global theory is in fact unstable to three-dimensional let alone two-dimensional disturbances. This is a common difficulty with global approaches. We have seen from the circular case that a p-unstable flow may be two-dimensionally stable, although it often appeared to be unstable to a particular kind of three-dimensional disturbance. We conjectured that such is usually the case for general streamline shapes, but the uncertainty is then even greater, there being no convenient normal mode theory with which to compare.

Secondly, even if a p-unstable flow is in fact unstable, we have not discovered the precise nature of the instability. Although the form of ϕ that minimizes J gives some indication as to the instability mechanism, this ϕ need not be a normal mode of the system. Physically, however, one might expect ϕ to be a reasonable approximation to an unstable mode, if the flow is indeed unstable to two-dimensional disturbances. What ϕ does indicate is the structural weakness in the given flow from which perturbations can extract energy.

The dependence of the stability problem on the vorticity gradient $d\omega/d\psi$ is manifest. This is not too surprising given the known behaviour of one-dimensional and circular flows. As we argued in §3, Arnol'd's first theorem may be considered as a generalization of Rayleigh's inflection-point criterion to arbitrarily shaped flows. The expression (1.3) for forced and dissipative flow affords further enlightenment. We know that the flow is stable when $d\omega/d\psi < 0$ everywhere. This corresponds to flow against the sense of the driving force. Physically therefore, the flow must be driven by some external surface stress τ. Such a configuration seems to be stable, possibly even to three-dimensional disturbances. When conversely, as is usually the case when rotational forces act, the flow is mainly in the same direction as the forcing, then $d\omega/d\psi > 0$. When the circulation around a given streamline is relatively small, $d\omega/d\psi$ is large and the flow is likely to be p-unstable. Instability seems physically plausible in this case, as a local perturbation may be able to extract energy from the forcing without commensurately increasing the dissipation rate. We conclude that flows are likely to be stable if driven in the main from their outer boundary, and that, when they are driven internally by a rotational force, instability may occur whenever geometrical constraints or boundary conditions cause a disproportionately low response to the forcing.

The above comments should be borne in mind when applying the results of this paper to an appropriate flow. It is worth re-emphasizing, however, that the class of flow to which they apply is wider than might appear at first sight, as the region of inertial domination described by (1.2) need not occupy the entire domain. Any flow, perhaps even one turbulent in places, may contain a portion bounded by a streamline inside which the conditions of this paper apply, and the stability of which may be determined by the methods we have described.

Although the emphasis of this paper has been on the two-dimensional theory, it is possible that §7 on axisymmetric flows is of the greatest practical importance. Such flows tend to have greater resistance to azimuthally dependent perturbations than do planar flows to disturbances in the third dimension. One possible application of the axisymmetric theory is to the levitation melting problem (e.g. Mestel 1982).

Finally, I would like to express my gratitude to Dr Shepherd, Dr Proctor and Professor Moffatt for some useful conversations and comments.

REFERENCES

ANDREWS, D. G. 1983 A conservation law for small-amplitude quasi-geostrophic disturbances on a zonally asymmetric basic flow. *J. Atmos. Sci.* **40**, 85–90.

ARNOL'D, V. I. 1965 Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Dokl. Akad. Nauk. SSSR* **162**, 975–978. (English transl. *Soviet Math.* **6**, 331–334.)

ARNOL'D, V. I. 1966 On an apriori estimate in the theory of hydrodynamic stability. *Izv. Vyssh. Uchebn. Zaved. Matematika* **54**, No. 5, 3–5. (English transl. *Amer. Math. Soc. Trans.* 2 **79**, 267–269 (1969).)

BAYLY, B. J. 1988 Three-dimensional centrifugal-type instabilities in inviscid two-dimensional flows. *Phys. Fluids* **31**, 56–64.

BLUMEN, W. 1968 On the stability of quasigeostrophic flow. *J. Atmos. Sci.* **25**, 929–931.

DRAZIN, P. & REID, W. 1981 *Hydrodynamic Stability*. Cambridge University Press.

FJØRTOFT, R. 1953 On the changes in the spectral distribution of kinetic energy for two-dimensional, nondivergent flow. *Tellus* **5**, 225–230.

GARABEDIAN, P. R. 1964 *Partial Differential Equations*, p. 412. Wiley.

JONES, C. A., MOORE, D. R. & WEISS, N. O. 1976 Axisymmetric convection in a cylinder. *J. Fluid Mech.* **73**, 353.

McINTYRE, M. E. & SHEPHERD, T. G. 1987 An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnol'd's stability theorems. *J. Fluid Mech.* **181**, 527–565.

MESTEL, A. J. 1982 Matnetic levitation of liquid metals. *J. Fluid Mech.* **117**, 27–43.

MESTEL, A. J. 1989 An iterative method for high Reynolds number flows with closed streamlines. *J. Fluid Mech.* **200**, 1–18.

MOFFATT, H. K. 1965 On fluid flow induced by a rotating magnetic field. *J. Fluid Mech.* **22**, 521.

MOFFATT, H. K. 1986 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations. *J. Fluid Mech.* **166**, 359–378.

MURZBACHER, E. 1970 *Quantum Mechanics*. Wiley.

RAYLEIGH, LORD 1880 On the stability, or instability, of certain fluid motions. *Proc. Lond. Math. Soc.* **10**, 4–13.

RICHARDSON, A. T. 1974 On the stability of a magnetically driven rotating fluid flow. *J. Fluid Mech.* **63**, 593–605.

TAYLOR, G. I. 1923 Stability of a viscous liquid contained between two rotating cylinders. *Phil. Trans. R. Soc. Lond.* A **223**, 289–343.